

Emergence and Reductionism: an awkward Baconian alliance

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Abstract

This article discusses the relationship between emergence and reductionism from the perspective of a condensed matter physicist. Reductionism and emergence play an intertwined role in the everyday life of the physicist, yet we rarely stop to contemplate their relationship: indeed, the two are often regarded as conflicting world-views of science. I argue that in practice, they compliment one-another, forming an awkward alliance in a fashion envisioned by the Renaissance scientist, Francis Bacon. Looking at the historical record in classical and quantum physics, I discuss how emergence fits into a reductionist view of nature. Often, a deep understanding of reductionist physics depends on the understanding of its emergent consequences. Thus the concept of energy was unknown to Newton, Leibnitz, Lagrange or Hamilton, because they did not understand heat. Similarly, the understanding of the weak force awaited an understanding of the Meissner effect in superconductivity. Emergence can thus be likened to an encrypted consequence of reductionism. Taking examples from current research, including topological insulators and strange metals, I show that the convection between emergence and reductionism continues to provide a powerful driver for frontier scientific research, linking the lab with the cosmos.

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I. INTRODUCTION: REDUCTIONISM AND EMERGENCE

Reductionism is the marvelous idea that as we take matter apart to its smallest constituents, and understand the laws and forces that govern them, we can understand everything. This bold idea traces back to Greek antiquity and has served as a key inspiration in the natural sciences, particularly physics, up to the present day. Emergence, by contrast, is the intriguing idea that as matter comes together, it develops novel properties and unexpected patterns of collective behavior ¹. This is something that scientists have long understood intuitively - we observe emergence all around us - from snowflakes floating on a cold day, the pull of a mundane refrigerator magnet, a flock of geese flying overhead, for those of us who have seen it, the magic of a levitating superconductor and life in all its myriad forms. These are all examples of natural science that are not self-evident linear extrapolations of the microscopic laws and which often require new concepts for their understanding.

Emergence and reduction are sometimes regarded as opposites. The reductionist believes that all of nature can be reduced to a “final theory”, a viewpoint expressed beautifully in Stephen Weinberg’s “Dreams of a final theory” [Weinberg: 1992]. Whereas reductionism is an ancient concept, the use of the word emergence in the physical sciences is a comparatively recent phenomenon, dating back to the highly influential article by Philip W. Anderson, entitled “More is Different” [Anderson: 1972]. In this highly influential work, Anderson put forward the idea that each level in our hierarchy of understanding of science involves emergent processes, and that moreover, the notion of “fundamental” physics is not tied to the level in the hierarchy. Yet despite the contrast between these two viewpoints, neither repudiates the other. Even the existence of a final theory does not mean that one can go ahead and calculate its consequences “ab initio”. Moreover, the existence of emergence is not a rejection of reductionism, and in no way implies a belief in forms of emergence which can never be simulated or traced back to their microscopic origins.

In this article I present a pragmatic middle-ground: arguing that reductionism and emergence are mutually complementary and quite possibly inseparable. Sometimes methods and insights gained from a reductionist view, including computational simulation, do indeed enable us to understand collective emergent behavior in higher-level systems. However, quite

¹ For a careful discussion of the definition of emergence in physics, see for example [Kivelson and Kivelson: 2016].

frequently an understanding of emergent behavior is needed to gain deeper insights into reductionism. Here, I will illustrate this viewpoint with examples from the historical record and also some current open challenges in condensed matter physics.

II. THE BACONIAN VIEW.

Four-hundred years ago, the Renaissance philosopher-scientist, Francis Bacon championed a shift in science from the top-down approach favored in classical times, to the empirically-driven model that has been so successful up to the current day. In 1620, Bacon wrote[Bacon: 1620]

“ There are and can exist but two ways of investigating and discovering truth. The one hurries on rapidly from the senses and particulars to the most general axioms, and from them, as principles and their supposed indisputable truth, derives and discovers the intermediate axioms. And this way is now in fashion. The other derives axioms from the senses and particulars, rising by a gradual and unbroken ascent, so that it arrives at the most general axioms last of all. This is the true way, but as yet untried.

Francis Bacon, *Novum Organum*, 1620

Baconian argues for an integrated experimental-theoretical approach to science, in which progress stems not from imposing the most general axioms, but by using experiment and observation without preconception, as guidance to arrive at the “most general axioms”. Bacon’s approach is not an abandonment of reductionism, but a statement about how one should use experiment and observation to arrive there. The Baconian approach however leaves room for surprises - for discoveries which are unexpected “collective” consequences of the microscopic world, consequences which often shed new light on our understanding of the microscopic laws of physics.

A. The incompleteness of Classical Mechanics

Modern education teaches classical mechanics as a purely reductionist view of nature, yet historically it remained conceptually incomplete until the nineteenth century, two hundred years after Newton and Leibnitz. Why? Because the concept of energy was unknown and

was not developed until heat was identified as an emergent consequence of random thermal motion.

Energy is most certainly a reductionist consequence of classical mechanics: if the force on a particle is the gradient of a potential, as for example in the motion of the planets, $\vec{F} = -\vec{\nabla}V = m d\vec{v}/dt$, then it follows that the energy $E = \frac{1}{2}mv^2 + V$ is conserved. Yet although Newton and Leibnitz understood the motion of the planets, understanding that was considerably sharpened by Lagrange and Hamilton, the concept of energy was unknown to them all.

Gottfried Leibnitz had intuitively identified the quantity mv^2 (without the half) as the *life force* (“*vis vivre*”) of a moving object, but he did not know that it was the conserved counterpart of momentum (“*quantitas motus*”). Lagrange[Lagrange: 1788] introduced the modern notation $T = \sum_j \frac{1}{2}m_j v_j^2$, with the factor of 1/2, and he certainly knew that $T + V$ was conserved provided both are time-independent, a point later formulated as consequence of time-translation symmetry by Emilie Noether in the 20th century². Yet still, the concept of energy had to wait a full two centuries after Newton. From a practical point-of-view, momentum is a vector quantity which is manifestly conserved in collisions, so that a macroscopic momentum can never dissipate into random microscopic motion. By contrast, energy as a scalar quantity inevitably transforms from manifest bulk kinetic energy, into microscopic motion. Without an understanding of heat, kinetic energy appears to vanish under the influence of friction.

In Munich in 1798, the Colonial American-born royalist, inventor and physicist, Benjamin Thompson (Count Rumford) carried out his famous experiment demonstrating that as a canon is bored, heat is produced. He wrote afterwards[Thompson: 1798] that

It appears to me to be extremely difficult, if not quite impossible, to form any distinct idea of any thing, capable of being excited and communicated in the manner the Heat was excited and communicated in these experiments, except be it MOTION.

Thompson’s identification of heat as a form of motion put an end to the “caloric” theory of heat as a fluid, clearing a conceptual log-jam that had prevented progress for two centuries,

² Lagrange writes in his treatise *Mécanique Analytique* “In effect the integral $T + V = \text{constant}$ follows when T and V have no t dependence” (“*En effect, l’intégrale $T + V = \text{const}$, ayant nécessairement lieu, puisque T & V sont fonctions sans t* ”)[Lagrange: 1788] .

From our 21st century hindsight, it seems almost inconceivable that several generations of physicists would miss energy conservation. Yet nothing in science is ever simple. It was certainly known to Louis Lagrange, and to William Hamilton after him, that the quantity $H = T + V$ is constant, but the notion of the universally conserved quantity we now call energy is completely absent from their theoretical treatises. Infact, though the word *energy* was most probably first introduced by Thomas Young in 1802[Young: 1807], the common usage of this concept had to wait until the middle of the 19th century.

The modern reductionist might argue that the early Newtonian physicists were just not reductionist *enough*! Perhaps, had they been so, they would have realized that the conservation law known for simple systems, should scale up the macroscopic. Yet, historically, until it was clear that heat was a form of random motion, this connection was not made. Newton, Leibnitz, Lagrange and Hamilton were the greatest minds of their generation, they believed fundamentally in the power of reductionism, yet they failed to make the link. Would a modern reductionist, without modern hindsight have fared any better? The fact is that the concept of energy was hidden from the most brilliant minds of the era and was not unlocked from its reductionist origins until physicists had understood one of the most remarkable emergent consequences of classical mechanics: heat. Classical mechanics thus provides a beautiful illustration of the intertwined relationship of reductionism and emergence.

B. Darwin-Maxwell-Boltzmann

Biologists trace the idea of *emergence* back to Charles Darwin's *Origin of the Species*, and the use of the term in science began in biology. Already, in the 19th century, scientists struggled with the relationship between emergence and reductionism. In the origin of the species, Darwin writes[Darwin: 1859]

whilst this planet has gone cycling on according to the fixed law of gravity, from so simple a beginning endless forms most beautiful and most wonderful have been, and are being evolved.

Charles Darwin, Origin of the Species (1859)

Here one glimpses a sense that emergence and reductionism are connected. Around the same time that Charles Darwin was writing his opus, a young James Clerk Maxwell was trying

to work out how Newton's laws could give rise to Saturn's rings. To describe the rings, Maxwell constructed what was in essence, an early model for his theory of atomic motion. In his prize essay on the theory of Saturn's rings, Maxwell[Maxwell: 1859] wrote

We conclude, therefore, that the rings must consist of disconnected particles; these may be either solid or liquid, but they must be independent. The entire system of rings must therefore consist either of a series of many concentric rings, each moving with its own velocity, and having its own systems of waves, or else of a confused multitude of revolving particles, not arranged in rings, and continually coming into collision with each other.

James Clerk Maxwell, On the stability of Saturn's Rings. Adams Prize (1859)

Maxwell understood that the properties of Saturn's rings were a collective consequence of collisions between its constituent particles. Later, when he moved from Aberdeen to London, he used the astronomic inspiration from Saturn's rings as a model to develop his molecular theory of gases. At a time where the concept of an atom was as controversial as modern string theory, his particulate model for Saturn's rings provided a valuable launching pad for his derivation of the kinetic theory of molecular motion.

Maxwell, and Boltzmann after him, realized the importance of the Baconian approach to science - and in particular, that the collective motion of particles required new statistical approaches, inspired by observation and experiment. Here's a quote from Boltzmann in the early 20th century[Cercignani: 2006]:

We must not aspire to derive nature from our concepts, but must adapt the latter to the former... Even the splitting of physics into theoretical and experimental is only a consequence of methods currently being used, and it will not remain so forever.

Ludwig Boltzmann, Theoretical Physics and Philosophical Problems (1905)

Boltzmann pioneered a reductionist explanation of thermodynamics and the field of statistical mechanics, yet it is clear he was strong believer in the importance of an empirically-based approach.

III. FROM THE ANGSTROM TO THE MICRON.

The vast discoveries in physics during the twentieth century, from the discovery of the structure of the atom, to relativity and quantum-mechanics, the successful prediction of anti-matter from relativistic quantum mechanics and the discovery of gauge symmetries that lie behind the standard model of particle physics, are a monumental tribute to the power of reductionism[Pais: 1986]. Today, the obvious extensions of this frontier lie in the puzzles of dark matter and dark energy, the observation of gravity waves, the confirmation of the Higg’s particle in the standard theory and string theory with its prediction of 10^{500} alternate multiverses[Weinberg: 2005]. The excitement of this frontier is widely shared with society, for instance, in Stephen Weinberg’s “Dreams of a Final Theory”, Brian Greene’s “Elegant Universe”[Greene: 1999] and Hawkings “Brief History of Time”[Hawking: 1988]. These expositions capture the beauty and romance of discovery, giving rise to a popular impression that the frontier of science is purely reductionist and that the frontiers lie at the sub-quark scale, the Planck mass and the first moments of the Big Bang.

Yet, this is only one element of today’s physics frontier: we need only look just beneath the limits of the optical microscope and classical engineering, to scales of order a micron to find remarkable emergent physics that we barely begin to understand. This is the view expounded by Philip W. Anderson (Fig. 1) in his highly influential article “More is different”[Anderson: 1972],

‘The behavior of large and complex aggregations of elementary particles, it turns out, is not to be understood in terms of a simple extrapolation of the properties of a few particles. Instead, at each level of complexity entirely new properties appear, and the understanding of the new behaviors requires research which I think is as fundamental in its nature as any other.’

Philip W. Anderson in “More is Different”.

Anderson’s article, and his subsequent writings helped to crystallize the idea of emergence in the physical sciences. The concept of emergence re-invigorated the field of solid state physics, prompting the field to redefine itself under the broader title “Condensed Matter Physics”.

The terrestrial counter-part to the multiverse of string theory is the periodic table. While there are only 92 stable elements, quantum mechanics and chemistry mean that each new



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*Philip. W. Anderson
from “More is Different”, 1967.*

FIG. 1. Philip W. Anderson. (Source: musicofthequantum.rutgers.edu) Anderson introduced the concept of emergence into condensed matter physics in the 1960’s in his influential “More is different” article.

compound provides a new universe of collective behavior. As we go from elements to binary, tertiary and quaternary compounds, out towards the organic molecules of life, the number of unique combinations exponentiates rapidly. It is this emergent multiverse that provides the backdrop for quantum materials, biology, life and all its consequences.

On the length-scale of atoms, an Angstrom (10^{-10}m) we understand pretty much everything about the motion of electrons and nuclei. This motion is described in terms of the Many-Body Schrödinger equation, which describes the system in terms of a wave, described by the many-body wavefunction $\Psi(1, 2, 3 \dots N)$, where $1, 2, \dots$ denote the co-ordinates of the particles. The squared magnitude of this wave provides the probability of finding the particles at their respective co-ordinates,

$$p(1, 2 \dots N) = |\Psi(1, 2 \dots N)|^2, \quad (1)$$

and in principle, with a few caveats, all the statistics of the particle motion, momentum, energy, the fluctuations, correlations and response can be determined from Ψ . One important aspect of this description, is its wave character, reflected by the phase of the wavefunction. When we add particles together, their waveforms overlap and interfere with each other, so that unlike classical systems, the probability distribution of the sum is not the sum of its

parts

$$|\Psi_A + \Psi_B|^2 \neq |\psi_A|^2 + |\psi_B|^2. \quad (2)$$

This is part of the answer to something chemists know intuitively: that when one combines elements together, the compound that forms is utterly different from a simple mixture of its components. The other important aspect is that the wavefunction depends on a macroscopically huge number of variables - classically, a system of N particles requires $3N$ position and momentum variables - a quantity that is itself huge; yet quantum-mechanically, the number of variables required to describe a wavefunction is an exponential of this huge number.

As we scale up from the Angstrom to the micrometer ($1\text{\AA} = 10^{-10}\text{m}$, $1\mu\text{m} = 10^{-6}\text{m}$), a mere four orders of magnitude, matter acquires qualitatively new properties. The particles come together together to form crystals: this we can understand classically. However, the electron waves that move throughout these immense periodic structures interfere with each other and this interference endows matter with remarkable new properties, hardness, rigidity, magnetism, metalicity, semi- and superconductivity, phase transitions, topology and much much more. To take an example proposed by Anderson[Anderson: 1972], on the scale of the nanometer, the motion of electrons in metallic gold is identical to that in niobium or tin. Yet on scales of a micron, electrons in niobium and tin correlate together into Cooper pairs to form superconductors that expel magnetic fields and levitate magnets. Niobium and tin are examples of low-temperature superconductors, requiring the extreme low temperatures of liquid helium to cool them to the temperatures where they conduct without resistance, but today physicists have discovered new families of “high temperature superconductors” that only require liquid nitrogen, and there is a dream that room temperature superconductivity might occur in hitherto undiscovered compounds. Yet superconductivity is just a beginning, for already by the micron, life develops. The organism *Mycoplasma Mycoides*, found in the human gut, forms self-reproducing cells of 250nm in diameter[Kuriyan et al.: 2013]. While we more-or-less understand the physics of Cooper pairs in periodic, equilibrium superconductors, we are far from understanding the emergent physics of life that develops on the same scale in aperiodic, non-equilibrium structures. This lack of understanding occurs despite our knowledge of the microscopic, many-body Schrödinger equation, and it is this realization that prompts us to appreciate emergence as a complimentary frontier[Laughlin et al.: 2000]. It prompts us to pose the question:

What are the principles that govern the emergence of collective behavior in matter?

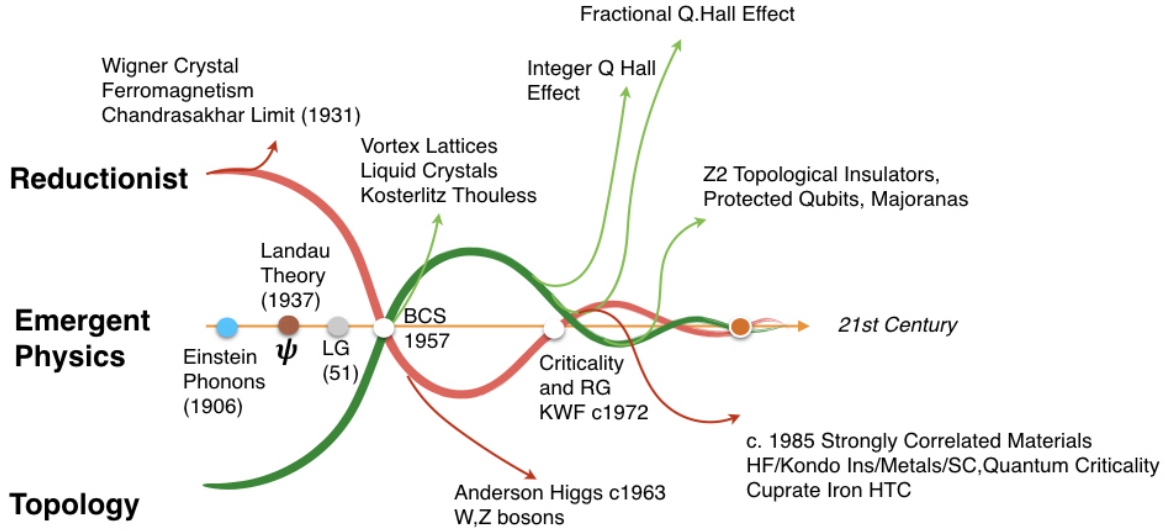


FIG. 2. Schematic time-line, illustrating developments in condensed matter physics over the past century. The three arrows show developments following a reductionist, emergent and topological track.

IV. A SELECTIVE HISTORY OF EMERGENCE AND REDUCTIONISM IN CONDENSED MATTER PHYSICS

Condensed matter physics is rife with historical examples of intertwined reductionism and emergence, with the one providing insights into the other (Fig. 2). One of the things we learn from these examples, is that fundamental physics principles are not tied to scale: that while insights from the cosmos influence our understanding in the lab, equally, understanding of emergent principles gleaned from small-scale physics in the lab has given us extraordinary new insights into the early universe and the sub-nuclear world.

To illustrate this interplay between emergence and reductionism, let us look at some examples. Quantum condensed matter physics arguably began with Albert Einstein’s 1906 proposal[Einstein: 1907] that the concept of quanta could be extended from light, to sound. In the previous year he had proposed the idea of quanta, or *photons* to interpret Planck’s

theory of black-body radiation[Einstein: 1905]. By proposing that light is composed of streams of indivisible quanta of energy $E = hf$, where $h=6.626 \times 10^{-34}$ Js is Planck’s constant and f is the frequency, Einstein was able to inject new physical insight into Planck’s earlier work, and using it, he could make the link between black-body radiation and the photo-electric effect. By 1906 he saw that he could take the idea one step further, proposing that analogous sound quanta occur in crystals. By treating a crystal as an “acoustic black-body”, Einstein was able to develop of theory of the low temperature specific heat capacity of diamond, as it drops below the constant value (“Dulong and Petit’s law”) predicted by classical equipartition. Einstein’s work in 1905 and 1906 are remarkable examples of high-grade phenomenology - driven by experiment and careful physical reasoning. Moreover, Einstein’s “phonons” as we now call them, are emergent quanta of the solid state: the result of the quantization of the collective motion of a macroscopic crystal.

Another other early idea of emergence in physics, is Landau’s order parameter theory of phase transitions, developed in 1937[Landau: 1937]. Here he introduced the key concepts of an order parameter and spontaneously broken symmetry: the main idea is that the development of order at a phase transition can be quantified in terms of *order parameter* ψ , which describes the development of a macroscopic property, such as a magnetization ($\psi = M$) or an electric polarization ($\psi = P$). With a very simple phenomenological theory, Landau showed how to use this concept to describe phase transitions, without reference to the microscopic origin of the order parameter or the mechanism by which it developed. In Landau’s theory, close to a second order phase transition, the dependence of the bulk free energy $F[\psi]$ on the order parameter is given by

$$F[\psi] = a(T - T_c)\psi^2 + b\psi^4 + O(\psi^6) \quad (3)$$

where a and b are positive constants, T is the temperature and T_c is the critical temperature. For $T > T_c$, the free energy is a minimum at $\psi = 0$, but for $T < T_c$, it develops two “broken symmetry” minima at $\psi = \pm[(a/2b)(T_c - T)]^{1/2}$ [Chaikin and Lubensky: 1995, Coleman: 2016]. The important point about Landau theory, is that it describes a universal property of matter near a phase transition, independently of the microscopic details of the material. Thirteen years later in 1950, Ginzburg and Landau[Ginzburg and Landau: 1950] showed how an more detailed version of Landau theory, or “Ginzburg Landau theory”, in which $\psi(x)$ is a complex order parameter with spatial dependence, could provide a rather complete

macroscopic description of superconductors accounting for the expulsion of magnetic flux and the levitation of magnets a half decade before the Bardeen Cooper Schrieffer (BCS) microscopic theory of the same phenomenon.

Yet condensed matter physics could not have developed without reductionism[Pais: 1986]. With the arrival of Heisenberg’s matrix mechanics in the 1920’s, it became possible to attempt a first-principles description of quantum matter. Suddenly, phenomena such as ferromagnetism that were literally impossible from a classical perspective, could be given a precise microscopic description, and these phenomena could be linked in a reductionist fashion to the equations of quantum mechanics. The idea that electrons are probability waves, described by Schrödinger’s equation, led to the notion of Bloch waves: electron waves inside crystals. The idea of antimatter, predicted by Paul Dirac using his relativistic theory of electrons[Dirac: 1931] had its direct parallel in condensed matter physics in Peierls’ and Heisenberg’s concept of “hole” excitations in semiconductors[Heisenberg: 1931, Hoddesson et al.: 1987]. Landau and Neél extended Heisenberg’s ideas of magnetism to predict antiferromagnetism, first observed in the 1950s while Wigner used reductionist principles to predict that electrons would form “Wigner crystals” at low densities, a remarkable result not confirmed until the 1980s. Quantum mechanics also enjoyed application in the new realm of astrophysics, most dramatically in Subrahmanyan Chandrasakhar’s theory of stellar collapse[Chandrasekhar: 1984]. By combining classical gravity with the statistical (quantum) mechanics of a degenerate fluid of protons and neutrons, Chandrasakhar was able to predict that beyond a critical mass, stars would become unstable and collapse. The critical Chandrasakhar mass M of a star,

$$M \approx M_P \left(\frac{M_P}{m_p} \right)^2 \quad (4)$$

is given in terms of the proton and the Planck mass, m_p and $M_P = \left(\frac{hc}{G} \right)^{\frac{1}{2}}$ respectively. Chandrasakhar’s formula, built on principles designed to understand the terrestrial statistical mechanics of electrons, is the first time that gravity and quantum mechanics come together in a single expression.

Yet the fully reductionist revolution of quantum mechanics ran out of steam when it came to understanding superconductivity: the phenomenon whereby metals conduct electricity without resistance at low temperatures. Some of the greatest minds of the first half of the 20th century, Bohr, Einstein[Sauer: 2008], Bloch, Heisenberg and Feynman[Schmalian:

2010] attempted microscopic theories of superconductivity, without success. In 1957, the reductionist and emergent strands of condensed matter physics, came together in a perfect storm of discovery, with the development of the Bardeen Cooper Schrieffer (BCS) theory of superconductivity[Bardeen et al.: 1957]. On the one hand, it required a reductionist knowledge of band theory and the interaction of electrons and phonons; it also took advantage of the new methods of quantum field theory, adapted from the theory of quantum electrodynamics by early pioneers such as Fröhlich, Gell-Mann and Hubbard. On the experimental front, it required the discovery of the Meissner effect: the expulsion of magnetic fields that occurs when a metal becomes superconducting; it also built strongly on the phenomenological ideas of London, Landau and Ginzburg, Pippard and Bardeen; finally, it required stripping the physics down to its bare minimum, in the form of a minimalist model now known as the “BCS model”. The important point is that rather than attempting a fully reductionist description of the combined electron-lattice and electron-electron interactions, which led to something far too complicated to be solved in one go, Bardeen, Cooper and Schrieffer captured the the combined effects of these phenomena in terms of a simple low-energy attractive interaction between pairs.

BCS theory had many further ramifications: pairing was generalized to the nucleus, where it led to an understanding of the stability of even-numbered nuclei; it led to the prediction of superfluidity in neutron stars and He-3. Most unexpectedly, it opened up new perspective on broken symmetry that inspired Anderson then Higgs and others to identify a mechanism for how gauge particles acquire mass that we now call the “Anderson-Higgs mechanism” [Anderson: 1963, Higgs: 1964]. At a time where particle physicists had almost abandoned field theory, the new success in superconductivity provided a case study of field theory in action that stimulated a resurgence of interest in field theory in particle physics, leading to Electro-weak theory[Witten: 2016]. Indeed, key elements of electro-weak theory can be understood as a simple two component spinorial extension of Landau Ginzburg theory, and from this perspective, the weak force in nuclear-particle physics can be understood as a kind of cosmic Meissner effect that expels the W and Z fields from our universe.

A second example of the intertwined nature of reductionism and emergence is provided by the theory of critical phenomenon, a revolution in understanding of phase transitions that occurred a decade after BCS theory, between 1965-1975[Domb: 1996]. From the sixties, physicists were increasingly aware of a failure in the classical theory of phase transitions,

based on the work of Van der Waals, Landau and others, which was unable to describe the observed properties of second order phase transitions. Experiments, plus Onsager’s solution to the two dimensional Ising model, showed that phase transitions were characterized by unusual, indeed, universal power-law behavior. For example, the magnetization of a ferromagnet below its critical temperature develops with a power-law $M \propto (T_c - T)^\beta$. Landau’s theory predicts $\beta = 1/2$, yet in three dimensional Ising ferromagnets, $\beta = 0.326\dots$. Moreover, the unusual critical exponents were found to occur in a wide variety of different phase transitions, exhibiting the phenomenon of “universality”.

To understanding this discrepancy required a revolution in statistical mechanics, involving new, high precision measurements of phase transition, it meant borrowing methods that had been developed to control or “renormalize” divergences in particle physics, but it also involved developing new ideas about how physics changes and scales with size. Today these ideas are captured by a “scaling equation” that describes the evolution of a Hamiltonian H with length-scale L . Schematically, such scaling equations are written as

$$\frac{\partial H}{\partial \text{Log}[L]} = \beta[H], \quad (5)$$

where H is the Hamiltonian, L represents some kind of minimum cut-off length-scale to which the Hamiltonian applies and $\beta[H]$, the function that describes how $H[L]$ depends on length scale is called the “beta function”. The culmination of this work in Fisher and Wilson’s “epsilon expansion”, showed how to calculate scaling behavior using a beautiful innovation of following physics as a function of dimension d [Wilson and Fisher: 1972]. Remarkably, for the simplest models, the classical theories of phase transitions worked in dimensions above $d = 4$. Wilson and Fisher showed that a controlled expansion of the critical properties could be developed in terms of the deviation from four dimensions $\epsilon = 4 - d$. The Fisher Wilson theory is a theory of an emergent phenomenon, yet it draws on methodologies from reductionist quantum field theory.

V. TWO EXAMPLES FROM CURRENT PHYSICS

The convective exchange of reductionist and emergent perspectives continues to drive current developments in condensed matter physics. I’d like to touch on two active examples: research into *topological* properties of quantum matter, and the mystery posed by

the discovery of classes of phase transitions at absolute zero, which radically transform the electrical properties of conductors into *strange metals*.

A. A topological connection

One of the most remarkable developments has been the discovery of a topological connection to emergence[Hasan and Kane: 2010, Moore: 2010]. Topology describes global properties of geometric manifolds that are unchanged by continuous deformations. For instance, a donut can be continuously deformed into a one-handled mug: the presence of the hole, or the handle is topologically protected and we say they have the same topology. Mathematics links the differential geometry of two dimensional manifolds to the topology

$$\text{differential geometry} \leftrightarrow \text{topology}$$

via the “Gauss Bonet” theorem,

$$\frac{1}{4\pi} \int \kappa dA = (1 - g)$$

which relates the area integral of the curvature to the number of handles or the *genus* g of the surface. Topology is a kind of mathematical emergence: a robust property that depends on the global properties of a manifold.

The rise of topology in condensed matter physics involved a marvelously tortuous path of discovery. While the microscopic physics is a reductionist consequence of the band-theory of insulators developed in the 1930s, the discovery of a topological connection had to await another half-century, culminated in the discovery of a new class of band insulator, the “topological insulator”. One of the remarkable properties of topological matter is that the surface remains metallic. The 2016 Nobel prize in physics to Haldane, Kosterlitz and Thouless was awarded for their early contributions to the topology in condensed matter physics.

Topological structures in physics can develop in both real space and in momentum space. An example of the first kind of topology, are vortices in a superfluid. In a superfluid, the phase $\phi(x)$ of the complex order parameter $\psi(\vec{x}) \propto e^{i\phi(x)}$ is a smooth function of position and in passing around a closed path the order parameter must change smoothly and come back to itself, so that the change in the phase must be an integer multiple n of 2π , $n \times 2\pi$.

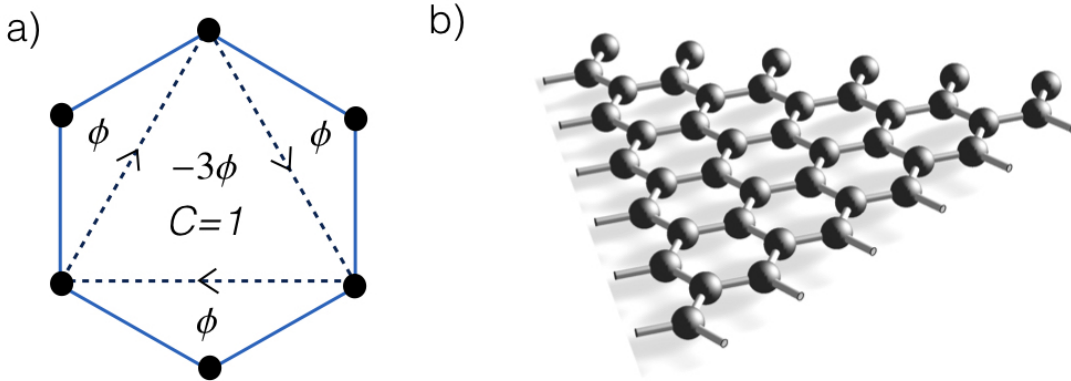


FIG. 3. (a) Haldane's tight-binding model after [Haldane: 1988] on a honeycomb lattice, used to show that topological Chern insulators can form without a net magnetic field. (b) Graphene, which together with Haldane's model, provided stimulus for the discovery of topological insulators.

The integer n describes the quantization of circulation in a superfluid, first predicted by Onsager and Feynman.

A second-kind of topology involves the wavefunction of electrons, in which a non-trivial topological configuration constitutes a new kind of “topological order”

differential geometry of the wavefunction \leftrightarrow topological order

Topological order is distinct from broken symmetry and it manifests itself through the formation of gapless surface or edge (2D) excitations around the exterior of an otherwise insulating state. The first example of such topological order is the quantization of the Hall constant in two dimensional electron gases, according to the relationship

$$\rho_{xy} = \frac{1}{\nu} \frac{h}{e^2}$$

where the Hall resistivity, $\rho_{xy} = V_H/I$ is the ratio of the transverse Hall voltage V_H to the current I and ν , an integer associated with the topology of the filled electron bands[Thouless et al.: 1982]; one of the manifestations of this effect, is the formation of ν “edge states” which propagate ballistically around the quantum Hall insulator.

Microscopically, this topology is determined by way the phase of the electron wavefunction twists through momentum space, which is given by a quantity called the “Berry connection”

associated with the filled electron bands, given by

$$\vec{\mathcal{A}}(\mathbf{k}) = -i \sum_{m=1,N} \langle u_{m,\mathbf{k}} | \nabla_{\mathbf{k}} | u_{m,\mathbf{k}} \rangle,$$

where $u_{m,\mathbf{k}}$ is the Bloch wavefunction of the m th filled electron band at momentum \mathbf{k} . The Berry connection $\vec{\mathcal{A}}(\mathbf{k})$ plays the role of an emergent vector potential: a momentum-space analog of the electromagnetic field. The corresponding magnetic flux, or “Berry curvature” $\kappa_{\mathbf{k}} = \vec{\nabla} \times \vec{\mathcal{A}}(\mathbf{k})$ plays the same role as the curvature in the Gauss-Bonnet theorem, and the integral of this curvature over momentum space gives the integer “Chern number”,

$$\nu = \frac{1}{2\pi} \int \kappa_{\mathbf{k}} d^2k. \quad (6)$$

Later in the 1980s, Duncan Haldane showed that such topological order could occur without a net magnetic field[Haldane: 1988]. Haldane’s 1987 theoretical model had a honeycomb structure (Fig. 3a.). Fifteen years later, the discovery of a 2D carbon structure “graphene”, with an uncanny resemblance to Haldane’s model, inspired Charlie Kane and Eugene Mele[Kane and Mele: 2005] to propose that topological order would develop in graphene without any magnetic field (Fig. 3b.). The key to their idea was “spin-orbit” coupling - an internal magnetic coupling between the spin and orbital motion of electrons. Kane and Mele recognized that spin-orbit coupling allows spin-up electrons to create a magnetic field for spin-down electrons, and vice versa, create two a quantum Hall effect for spin- up and spin-down electrons. The resulting edge states carry spin, forming an early version of the modern topological insulator, the “spin-Hall insulator”.

Although the spin-orbit coupling in real graphene turned out to be too weak to give rise to a topological insulator, the idea held and was confirmed by experiment[Bernevig et al.: 2006, König et al.: 2007] and later generalized to the three dimensional topological insulators[Fu and Kane: 2007, Fu et al.: 2007, Roy: 2009]. The current view is that spin-orbit coupling changes the topology of an insulator by inducing a crossing between the unoccupied conduction and occupied valence bands. Such crossings can only take place at certain allowed high symmetry points in momentum space defined by the crystal symmetry, and when they do, they change the topology. Like the braiding of a ribbon, where an odd number of twists produces a non-trivial configuration or Möbius strip, in insulators, an odd number of band crossings leads to a “strong topological insulator” (STI) with conducting surface states.

From a fully reductionist viewpoint, one might wonder why the topological revolution did not occur along with the development of electron band theory, from which it can be deduced. Indeed, one of the early pioneers of band theory, the co-inventor of the transistor, William Shockley[Shockley: 1939], came remarkably close. Yet new emergent principles, while traceable back to their microscopic origins, required the experimentally-inspired development of new concepts. We see here a close analogy with the 200 year delay in the discovery of energy as a consequence of Newtonian mechanics.

B. Strange Metals

As a counterpoint to the discovery of topological insulators, I'd like to say a little about how our understanding of metals appears to be on the verge of radical change. The foundations of the modern theory of metals were established not long after the discovery of the electron, at the turn of the 20th century by Paul Drude. One of the main ideas of Drude's theory of metals, is that electrons diffuse through a metal, due to their scattering off imperfections and vibrations. The resulting "transport relaxation time" τ_{tr} governs most aspects of the electron transport. The arrival of quantum mechanics in the 1920s led to a major upheaval in the understanding of the electron fluid. In particular, electrons, as identical quantum particles, were found to obey the Pauli Exclusion principle, which prevents more than one of them occupying the same eigenstate. This individualism causes electrons to fill up momentum space to higher and higher momentum states up to some maximum momentum, the Fermi momentum. The occupied states at this maximum momentum define a Fermi surface in momentum space, and almost all the action in a metal involves electrons at the Fermi surface.

Yet when the dust of quantum mechanics settled, Drude's picture had survived almost intact: in particular, the concept of a transport relaxation time could be extended to describe the scattering of electrons at the Fermi surface by disorder and mutual interactions, leading back to Drude's diffusive electron transport picture. One of the consequences of this robustness, is that one can measure the resistivity, Hall constant and the dependence of its resistivity on a magnetic field, the so-called "magneto-resistance", to check if these quantities scale with the scattering time τ_{tr} in the way predicted by Drude theory. Although the rate at which electrons scatter is temperature dependent, various ratios appearing in the transport

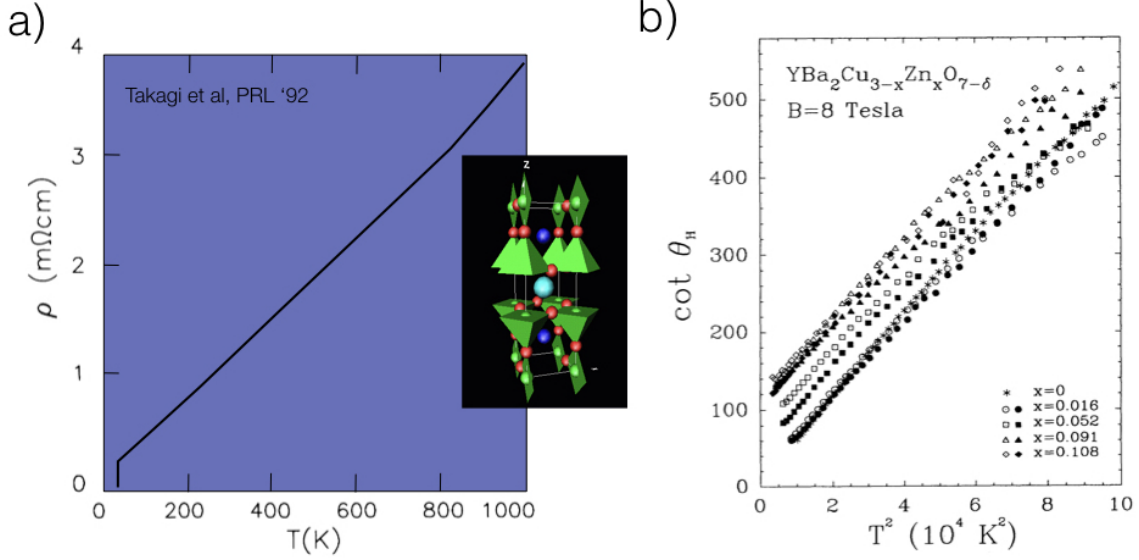


FIG. 4. Strange metals. (a) Linear resistivity of the high temperature superconductor $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ ($x=0.15$) (adapted with permission from H. Takagi *et al*, Phys. Rev. Lett. 69, 2975 (1992) [Takagi *et al.*: 1992]), showing the remarkable linear resistivity up to 1000K, indicating that the electrical current relaxation rate $\Gamma_{tr} \propto T$ is proportional to the temperature. (b) Quadratic temperature dependence of the Hall angle in a cuprate superconductor (reprinted with permission from T. R. Chien, *et al.*, Phys. Rev. Lett. 67, 2088 (1991)[Chien *et al.*: 1991]), indicating that Hall currents in these strange metals exhibit a decay rate $\Gamma_H \propto T^2$. The appearance of two relaxation time-scales in a simple conductor poses a challenge to our current understanding of metals.

theory are independent of the scattering rate and become temperature independent. One well-known consequence of Drude theory, is that the cancellation between the scattering rate associated with the Lorentz force cancels with the scattering rate due to the electric force, so that the ratio of the two, determined by the Hall constant $R_H = V_H/I$ is temperature independent. Another consequence is a scaling law known as Kohler's law. In Drude theory the resistivity R is proportional to the scattering rate $R \propto \tau_{tr}^{-1}$, whereas the magneto-resistivity grows with the square of the angle of deflection (Hall angle) of the electrical current in a field, $\Delta R/R \propto \Theta_H^2$. Now the Hall angle depends on the product of the cyclotron precession frequency and the scattering rate, $\theta_H = \omega_c \tau_{tr}$, so that $\Delta R/R \propto \theta_H^2 \sim \tau_{tr}^2$, which when combined with the resistivity, leads to Kohler's rule $\Delta R/R \propto R^{-2}$. This scaling relation works

remarkably well for a wide range of simple metals, vindicating Drude’s theory.

Of course, Quantum mechanics does have radical consequences for metals. For example, disorder can cause electron waves to *Anderson localize*[[Abrahams et al.: 1979](#), [Anderson: 1958](#)], completely stopping electron diffusion to produce an insulator. The many-body version of this phenomenon, *many-body localization*[[Nandkishore and Huse: 2015](#)] is of great current interest. Another radical consequence that I want to discuss now, is the formation of *strange metals*. Over the past three decades, experiments have revealed a new class of “strange metal” which deviates from Drude theory in a qualitative way. This unusual metallic behavior tends to develop in metals that are close to instability. When the interactions are increased inside a metal, through the effect of pressure, chemistry or external fields, the metal can become unstable, giving rise to *Quantum Phase transition* into an ordered state, such as magnetism. Such instabilities, occur at a absolute zero, where there are no thermal fluctuations to drive a phase transition. Instead, the phase transition is driven by quantum zero point fluctuations, and it is thought that these fluctuations play a role in transforming the electron fluid causing the resulting conductor to deviate qualitatively from Drude behavior.

The most famous strange metals are the high temperature cuprate superconductors[[Chien et al.: 1991](#), [Takagi et al.: 1992](#)], but similar behavior is also seen in their low temperature cousins, the family of heavy electron superconductors known as “115” superconductors[[Nakajima et al.: 2004](#)], and most recently, in artificially constructed two dimensional electron gases[[Mikheev et al.: 2015](#)], which are not superconducting. High temperature cuprate superconductors lose their resistance at temperatures as high as 90K, high enough that one can use liquid nitrogen to drive them superconducting. But above these temperatures, they are equally remarkable, for they exhibit a resistivity that is linear up to very high temperatures $R(T) \propto T$ (Fig. 4a.). Infact, this linearity can be traced back to a Drude scattering rate $\Gamma_{tr} = \tau_{tr}^{-1}$ that is proportional to the temperature, given approximately by $\Gamma_{tr} \sim \frac{k_B T}{\hbar}$. The time-scale $\tau_{tr} \sim \frac{\hbar}{k_B T}$ is sometimes called the “Planck time”, because it is the time scale derived from combining the energy-time uncertainty relationship $\Delta E \Delta \tau \sim \hbar$ with the Boltzmann energy $\Delta E \sim k_B T$. This simple scaling of the scattering rate with the temperature is very unusual, and in a typical metal the scattering rate has a much more complicated dependence on temperature, on disorder and on the coupling to vibrations of the crystal. Perhaps the strangest aspect of these metals, is their departure from

Drude behavior in a magnetic field because the scattering response to the Lorentz force, measured in a magnetic field is qualitatively different to the response to a pure electric field. Whereas the linear resistivity gives a scattering rate Γ_{tr} proportional to temperature, the magneto-resistivity and Hall resistivity give a scattering rate that is quadratic in the field $\Gamma_H \propto \theta_H^{-1} \propto T^2$ (Fig. 4b). Summarizing

$$\Gamma_{tr} \propto T, \quad \Gamma_H \propto \frac{T^2}{W},$$

where W is a scale that governs the decay of Hall currents. The presence of these qualitatively different scattering rates leads to a strongly temperature dependent Hall constant, and a “modified Kohler’s rule”, whereby the magneto-transport scales with the square of the Hall angle, rather than the square of the conductivity,

$$R(T) \propto \tau_{tr}^{-1}, \quad \frac{\Delta R}{R} \propto \theta_H(T)^2$$

This behavior is not unique to cuprate superconductors, and it has also been observed in certain heavy fermion superconductors[Nakajima et al.: 2004], which are low temperature cousins of the cuprate superconductors, and in low dimensional oxide interfaces[Mikheev et al.: 2015], but which have quite different microscopic chemistry and structure. These results taken together suggest that a fundamentally new kind of metal has been discovered, one that may require a new conceptual framework for interacting electrons. Unlike the new developments in our understanding of insulators, the emergent framework for understanding strange metals is still very much in the early days of discovery.

VI. CONCLUSION

This article has illustrated examples of emergence in condensed matter physics, seeking to highlight the close interdependence of a reductionist and emergent approach. Perhaps the most exciting aspect of this linkage, is that it may provide a way to accelerate the way we solve major problems in physics and the Natural sciences. While reductionism provides the mathematics and the computational tools to tackle complex problems and gain new insight into emergence, at the same time, it is likely that the importance of understanding of physics in the lab, particularly emergent physics between the Angstrom and the Micron, will, as it has in previous centuries, yield important insights into our reductionist understanding.

As in previous generations, condensed matter physicists are looking to the tools of particle physics, such as the holographic principle[Zaananen et al.: 2015], to make new progress on the many body problem, while in a similar vein, particle physicists and cosmologists are looking to emergence and condensed matter for inspiration. One of the prevalent ideas for unifying gravity and quantum mechanics is that space-time itself may be an emergent property of quantum gravity on scales beyond the Planck length[Seiberg: 2006]. Another area of activity is the problem of dark matter. For example, recently Verlinde[Verlinde: 2016] has suggested that the dark matter problem may be a consequence of an emergent aspect of gravity in which the unseen gravitating force inside galaxies is not interpreted as a cloud of particles, but as a kind of gravitating condensate.

These developments tempt us to wonder whether quantum mechanics might today be in a situation strangely parallel to classical mechanics two centuries after Newton and Leibnitz. Could it be that just as classical mechanics was incomplete until 200 years after Principia, because one of its emergent consequences, heat, prevented an understanding of energy, that today, 90 years after Heisenberg, Schrödinger and Dirac, the full reductionist consequences of the quantum await a similar advance in our understanding of its emergent consequences?

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